Typical Lower $L^q$-dimensions of Measures for $q \leq 1$

Yun Shi *, Meifeng Dai
Nonlinear Scientific Research Center, Jiangsu University
Zhenjiang, Jiangsu,212013,P.R.China
(Received 7 June 2008, accepted 10 August 2008)

Abstract: For a probability measure $\mu$ on a compact subset of $\mathbb{R}^d$, the lower $L^q$-dimension of order $q \in \mathbb{R}$ is defined by

$$D_\mu(q) = \liminf_{r \searrow 0} \log I_\mu(r; q) \frac{\log \mu(B(x, r))^{q-1} d\mu(x)}{-\log r}.$$  

In [7], the typical behavior (in the sense of Baire’s category) of the $L^q$-dimension $D_\mu(q)$ for $q \geq 1$ is studied. In [8], L. Olsen gave us two conjectures about the typical behavior of the $L^q$-dimension $D_\mu(q)$ for $q \leq 1$. In this paper, we discuss the two conjectures.

Key words: multifractal; $L^q$-dimension; residual set

1 Statement of the result

Recently, the properties of probability measure, self-similar measure and image measure have been discussed. For related topics, please see [1], [2] and [3]. Here we focus on probability measure on a compact subset of $\mathbb{R}$. Let $K$ be a compact subset of $\mathbb{R}$. Below $B(x, r) = \{y \in \mathbb{R}^d \mid |y-x| < r\}$. For a probability measure $\mu$ on $K$, the lower $L^q$-dimension of $\mu$ is defined as follows. For $r > 0$ and a real number $q$, write

$$I_\mu(r; q) = \int_K \mu(B(x, r))^{q-1} d\mu(x).$$

The lower $L^q$-dimension of order $q$ is now defined by

$$D_\mu(q) = \liminf_{r \searrow 0} \log I_\mu(r; q) \frac{\log \mu(B(x, r))^{q-1} d\mu(x)}{-\log r}.$$  

The main significance of the $L^q$-dimensions is their relationship with the multifractal spectrum of $\mu$. For a probability measure $\mu$ on $\mathbb{R}^d$ (or on a general metric space), the local dimension of $\mu$ at the point $x$ is defined by

$$\dim_{\text{loc}}(x; \mu) = \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$  

We define the Hausdorff multifractal spectrum function, $f_\mu$, of $\mu$ as the Hausdorff dimension of the level sets of the local dimension of $\mu$, i.e. we put

$$f_\mu(\alpha) = \dim \{x \in \mathbb{R}^d \mid \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha\}, \quad \alpha \geq 0,$$

where dim denotes the Hausdorff dimension. Next, recall that the Legendre transform $\varphi^*$ of a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\varphi^*(x) = \inf_y (xy + \varphi(y))$. In the 1980s it was conjectured in the physics

*Corresponding author.  E-mail address: daimf@ujs.edu.cn

Copyright©World Academic Press, World Academic Union
IJNS.2009.04.15/224
literature [4,5] that for good measures the following result, relating the multifractal spectrum function \( f_\mu \) to the Legendre transform of the \( L^q \)-dimensions, holds: namely (1) that the \( L^q \)-dimensions coincide, and (2) that the multifractal spectrum function \( f_\mu \) coincide with the value of the Legendre transform of the \( L^q \)-dimensions, i.e.

\[
D_\mu(q) = \overline{D}_\mu(q),
\]

and

\[
\dim\{ x \in K | \lim_{r \downarrow 0} \frac{\log \mu( B( x, r ))}{\log r} = \alpha \} = D^*_\mu(\alpha) = \overline{D}^*_\mu(\alpha)
\]

for all \( q \in \mathbb{R} \) and \( \alpha \geq 0 \). This result is known as the Multifractal Formalism. During the 1990s there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of various classes of measures in Euclidean space \( \mathbb{R}^d \) exhibiting some degree of self-similarity have been computed rigorously, cf. [6] and the references therein.

In this paper we study the lower \( L^q \)-dimension of a typical measure in the sense of Baire. For a compact subset \( K \) of \( \mathbb{R}^d \), we denote the family of Borel probability measures on \( K \) by \( \mathcal{P}(K) \) and we equip \( \mathcal{P}(K) \) with the weak topology. We will now say that a typical probability measure on \( K \) has property \( P \), if the set of probability measures that do not have property \( P \), i.e. if the set

\[
\left\{ \mu \in \mathcal{P}(K) | \mu \text{ does not have property } P \right\}
\]

is of the first category with respect to the weak topology on \( \mathcal{P}(K) \). The typical behaviour of various other quantities related to multifractal analysis has been studied.

In [7], we found the lower \( L^q \)-dimension of a typical measure for \( q \geq 1 \), and the purpose of this paper is to study lower \( L^q \)-dimension of a typical measure for \( q \leq 1 \). To state this result we begin with a few definitions. For a subset \( E \) of \( \mathbb{R}^d \), we denote the lower box dimension of \( E \) and the upper box dimension of \( E \) by \( \dim_B(E) \) and \( \overline{\dim}_B(E) \), respectively, the reader is referred to [6] for the definitions of the box dimensions. Also, for a subset \( K \) of \( \mathbb{R}^d \) and \( x \in K \) we define the lower local box dimension of \( K \) at \( x \) and the upper local box dimension of \( K \) at \( x \) by

\[
\dim_{B,loc}(x, K) = \lim_{r \downarrow 0} \dim_B( K \cap B(x, r) ),
\]

and

\[
\overline{\dim}_{B,loc}(x, K) = \lim_{r \downarrow 0} \overline{\dim}_B( K \cap B(x, r) ),
\]

respectively. We can now state the result from [7] giving the lower \( L^q \)-dimension of a typical measure for \( q \geq 1 \) and two conjectures about the lower \( L^q \)-dimension of a typical measure for \( q \leq 1 \) from [8].

**Theorem 1** (See[7].) Let \( K \) be a compact subset of \( \mathbb{R}^d \). Write

\[
s = \inf_{x \in K} \dim_{B,loc}(x, K),
\]

\[
\overline{s} = \inf_{x \in K} \overline{\dim}_{B,loc}(x, K),
\]

\[
s = \dim_B(K).
\]

Observe that \( s \leq \overline{s} \leq s \). A typical measure \( \mu \in \mathcal{P}(K) \) satisfies the following

\[
\pi(1 - q) \leq D_\mu(q) \leq s(1 - q)
\]

for all \( q \geq 1 \).

**Conjecture 1** (See[8].) Let \( K \) be a compact subset of \( \mathbb{R}^d \). A typical measure \( \mu \in \mathcal{P}(K) \) satisfies the following

\[
D_\mu(q) = 0
\]

for all \( q \in [0, 1] \).

_II NS email for contribution: editor@nonlinearscience.org.uk_
Conjecture 2 (See [8].) Let $K$ be a compact subset of $\mathbb{R}^d$. A typical measure $\mu \in \mathcal{P}(K)$ satisfies the following

$$D_\mu(q) = 0$$

for all $q < 0$.

Next, we give a positive answer to Conjecture 1 and a negative answer to Conjecture 2.

2 The proof of the results

2.1 The proof of Conjecture 1

Write

$$\Gamma = \{ \mu \in \mathcal{P}(K) \mid D_\mu(q) = 0 \text{ for all } q \in [0, 1] \}.$$

In order to prove Conjecture 1, it suffices to prove that the set $\Gamma$ is residual.

It is well-known that (cf., for example, [9, p. 51, Theorem 6.8]) the weak topology on $\mathcal{P}(K)$ is induced by the metric $L$ on $\mathcal{P}(K)$ defined as follows. Let $\text{Lip}(K)$ denote the family of Lipschitz functions $f : K \to \mathbb{R}$ with $|f| \leq 1$ and $\text{Lip}(f) \leq 1$ where $\text{Lip}(f)$ denotes the Lipschitz constant of $f$. The metric $L$ is now defined by

$$L(\mu, \nu) = \sup_{f \in \text{Lip}(K)} \left| \int f d\mu - \int f d\nu \right|$$

for $\mu, \nu \in \mathcal{P}(K)$. We will always equip $\mathcal{P}(K)$ with the metric $L$ and all balls in $\mathcal{P}(K)$ will be with respect to the metric $L$, i.e. if $\mu \in \mathcal{P}(K)$ and $r > 0$ we will write $B(\mu, r) = \{ \nu \in \mathcal{P}(K) \mid L(\mu, \nu) < r \}$ for the ball with the centre at $\mu$ and radius equal to $r$. For $x \in K$ and $r > 0$, define $f_{x,r} : K \to \mathbb{R}$ by

$$f_{x,r}(t) = \begin{cases} r & \text{if } |x - t| \leq r; \\ -|x - t| + 2r & \text{if } r < |x - t| < 2r; \\ 0 & \text{if } 2r \leq |x - t|. \end{cases}$$

Observe that if $r \leq 1$, then $f_{x,r}$ is Lipschitz with $|f_{x,r}| \leq 1$ and $\text{Lip}(f_{x,r}) = 1$. In particular, this implies that if $r \leq 1$, then

$$\left| \int f_{x,r} d\mu - \int f_{x,r} d\nu \right| \leq L(\mu, \nu)$$

for all $\mu, \nu \in \mathcal{P}(K)$.

Next we prove that the set $\Gamma$ is residual. It clearly suffices to construct a set $M \subseteq \mathcal{P}(K)$ satisfying the following three conditions:

1. $M \subseteq \Gamma$;
2. $M$ is dense in $\mathcal{P}(K)$;
3. $M$ is $G_\delta$.

For a positive integer $n$ write

$$\Lambda_n = \left\{ \lambda \in \mathcal{P}(K) \mid \lambda(\{x_0\}) \geq \frac{1}{n} \text{ for some } x_0 \in K \right\}.$$

Next put

$$G_n = \bigcup_{\lambda \in \Lambda_n} B(\lambda, \frac{1}{4n^{n+1}}),$$

and define the set $M \subseteq \mathcal{P}(K)$ by

$$M = \bigcap_{n \geq m} G_n.$$

Below we show that the set $M$ has the following three properties: (1) $M \subseteq \Gamma$, (2) $M$ is dense in $\mathcal{P}(K)$, and (3) $M$ is $G_\delta$. In [7] it has been proved that $M$ is dense in $\mathcal{P}(K)$, and the set $M$ is $G_\delta$, and it thus suffices to show that $M \subseteq \Gamma$.
Let $\mu \in M$ and fix a positive integer $m$. Since $\mu \in M$, there exists $n \geq m$ and a measure $\lambda \in \Lambda_n$ such that $L(\mu, \lambda) \leq \frac{1}{4n^m}$. Also, since $\lambda \in \Lambda_n$, we can find a point $x_0 \in K$ with $\lambda(\{x_0\}) \geq \frac{1}{n}$. For brevity write $r_n = \frac{1}{n^m}$. Now observe that for all $x \in B(x_0, \frac{r_n}{3})$, we have

$$
\mu(B(x, r_n)) = \int 1_{B(x, r_n)} d\mu \\
\geq \int \frac{f_{x_0, r_n}}{\frac{r_n}{3}} d\mu \\
\geq \frac{3}{r_n} \left( -L(\lambda, \mu) + \int f_{x_0, r_n} d\lambda \right) \\
\geq \frac{3}{r_n} \left( -\frac{1}{4n^{m+1}} + f_{x_0, r_n}(x_0) \lambda(\{x_0\}) \right) \\
\geq \frac{3}{r_n} \left( -\frac{1}{4n^{m+1}} + \frac{r_n}{3} \cdot \frac{1}{n} \right) \\
= \frac{1}{4n}.
$$

Hence

$$
T_1 = \int_{B(x_0, \frac{r_n}{3})} \mu(B(x, r_n))^{q-1} d\mu(x) \\
\leq \int_{B(x_0, \frac{r_n}{3})} \left( \frac{1}{4n} \right)^{q-1} d\mu(x) \\
= \left( \frac{1}{4n} \right)^{q-1} \mu(B(x_0, \frac{r_n}{3})) \\
\leq \left( \frac{1}{4n} \right)^{q-1}
$$

for all $q \in [0, 1]$.

Let $E = K \setminus B(x_0, \frac{r_n}{3})$, it clearly shows that $E$ is a compact subset of $K$. We can thus choose finitely many balls $B(x_1, \frac{r_n}{3}), \ldots, B(x_N, \frac{r_n}{3})$ such that $E \subseteq \bigcup_{i=1}^{N} B(x_i, \frac{r_n}{3})$. Put $E_1 = B(x_1, \frac{r_n}{3})$ and $E_i = B(x_i, \frac{r_n}{3}) \setminus \bigcup_{i=1}^{N-1} B(x_j, \frac{r_n}{3})$ for $i = 2, \ldots, N$. Next observe that if $x \in E_i$, then $E_i \subseteq B(x, r_n)$. We conclude from this that

$$
T_2 = \int_{E} \mu(B(x, r_n))^{q-1} d\mu(x) \\
\leq \sum_{i} \int_{E_i} \mu(B(x, r_n))^{q-1} d\mu(x) \\
\leq \sum_{i} \int_{E_i} \mu(E_i)^{q-1} d\mu(x) \\
= \sum_{i} \mu(E_i)^q.
$$

As $q \in [0, 1]$, the function $t \to t^q$ is concave, and Jensen’s inequality therefore implies that

$$
T_2 \leq \sum_{i} \mu(E_i)^q = N \sum_{i} \frac{1}{N} \mu(E_i)^q \leq N \left( \sum_{i} \frac{1}{N} \mu(E_i) \right)^q = \frac{1}{N^{q-1}} \left( \sum_{i} \mu(E_i) \right)^q
$$

$$
= \frac{1}{N^{q-1}} \mu\left( \bigcup_{i} E_i \right)^q \leq \frac{1}{N^{q-1}}.
$$

Hence

$$
I_\mu(r_n; q) = \int_{K} \mu(B(x, r_n))^{q-1} d\mu(x) \\
= T_1 + T_2 \\
\leq \left( \frac{1}{4n} \right)^{q-1} + \frac{1}{N^{q-1}}
$$
for all $q \in [0, 1]$. This implies that

$$
D_{\mu}(q) = \liminf_{r \to 0} \frac{\log I_{\mu}(r, q)}{-\log r} \leq \liminf_{n \to \infty} \frac{\log I_{\mu}(r_n, q)}{-\log r_n} \leq \liminf_{n \to \infty} \frac{\log \left(\frac{1}{2n} q^{-1} + \frac{1}{N q^{-1}}\right)}{-n \log n} = 0.
$$

This completes the proof of $M \subseteq \Gamma$.

### 2.2 A counterexample of Conjecture 2

For a probability measure $\mu$ on $\mathbb{R}^d$ and a real number $q$ write

$$
M_{\mu}(r; q) = \inf_{(B(x, r)), \text{is a centered cover of supp } \mu} \sum_i \mu(B(x_i, r))^q,
$$

where supp $\mu$ denotes the support of $\mu$. Now put

$$
\tau_{\mu}(q) = \liminf_{r \to 0} \frac{\log M_{\mu}(r; q)}{-\log r},
$$

$$
\tau_{\mu}(q) = \limsup_{r \to 0} \frac{\log M_{\mu}(r; q)}{-\log r}.
$$

The numbers $\tau_{\mu}(q)$ and $\tau_{\mu}(q)$ may clearly be thought of as box-counting versions of the $L^q$-dimensions $D_{\mu}(q)$ and $\overline{D}_{\mu}(q)$. Indeed, if $\mu$ is a doubling measure, we have $\tau_{\mu}(q) = D_{\mu}(q)$ and $\tau_{\mu}(q) = \overline{D}_{\mu}(q)$ for all $q$ (cf.[10]). The function $t \to t^q$ is still concave for all $q < 0$. Analogously, we can get

$$
\tau_{\mu}(q) \geq (1 - q) \dim_B(\text{supp } \mu),
$$

from Jensen’s inequality. In particular, this inequality shows that

$$
\tau_{\mu}(q) \geq \dim_B(\text{supp } \mu),
$$

(1)

for all $q < 0$. However, it is known (cf.[11]) that if $K$ is a compact subset of $\mathbb{R}^d$, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies:

$$
\text{supp } \mu = K.
$$

(2)

Combining (1) and (2) shows that if $K$ is a compact subset of $\mathbb{R}^d$, then a typical measure $\mu \in \mathcal{P}(K)$ satisfies:

$$
\tau_{\mu}(q) \geq \dim_B(\text{supp } \mu) = \dim_B(K),
$$

for all $q < 0$.

Let $K = [0, 1]$ and a typical and doubling measure $\mu \in \mathcal{P}([0, 1])$, then from above we have $\tau_{\mu}(q) \geq \dim_B([0, 1]) = 1$ for all $q < 0$. So we obtain $D_{\mu}(q) \geq 1$ for all $q < 0$, which contradicts with Conjecture 2.

### Acknowledgements

Research is supported by the National Science Foundation of China (10671180), the Education Foundation of Jiangsu Province (08KJB110003) and Jiangsu University (05JDG041).

IINS homepage: http://www.nonlinearscience.org.uk/
References


IJNS email for contribution: editor@nonlinearscience.org.uk